Spatial optical solitons play a fundamental role in the dynamics of nonlinear beams, and an accurate description of properties such as oblique propagation and mutual interactions is essential. Solitons that propagate at modest or large angles relative to the reference longitudinal direction, or to each other, experience a type of nonparaxiality that can be accurately described by a nonlinear Helmholtz equation (NHE). For a single soliton beam that coincides with longitudinal components and thus of soliton interactions at arbitrary angles.

Some optical contexts, such as intense self-focusing, give rise to a more general type of nonparaxiality. Paraxiality is commonly defined through a small parameter, \( \kappa = w_0^2/4L_D^2 \), where \( w_0 \) is the beam width, \( L_D = kW_0^2/2 \), and \( k = n_0\omega/c \). Order-of-magnitude analysis, based on \( \kappa \), then yields leading corrections to a paraxial wave equation. A near-paraxial beam, well described by scalar electric-field and refractive-index distributions, if it is considered in a reference frame rotated by \( \theta \), acquires an effective transverse velocity \( V \), but the beam itself remains intrinsically scalar in character. The usual paraxial condition, \( \kappa = 0 \), is still preserved, but now \( 2K V^2 = \tan^2 \theta \) can assume arbitrarily large values. In this Letter, the presence of this type of potentially dominant nonparaxial correction is demonstrated through exact solution of the NHE. Helmholtz-type nonparaxiality alone is shown to result in nontrivial modifications to soliton propagation characteristics.

The equivalence of the nonparaxial nonlinear Schrödinger equation and the appropriate NHE was recently noted. It permits identification of nonparaxial generalizations of conventional soliton theory as exact analytical Helmholtz bright soliton solutions. Physical interpretations and analytical properties of Helmholtz bright solitons have been described; they permit the development and testing of new nonparaxial beam propagation techniques.

Here we report, for the first time to our knowledge, a general Helmholtz dark-soliton solution for a defocusing Kerr nonlinearity. The conditions for experimental achievement of optical dark solitons are well known, and our theoretical predictions are expected to be directly observable. To highlight the modifications to paraxial theory, we solve the equivalent defocusing nonparaxial nonlinear Schrödinger equation:

\[
\kappa \frac{\partial^2 u}{\partial \xi^2} + i \frac{\partial u}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 u}{\partial \zeta^2} - |u|^2 u = 0 ,
\]

where \( \zeta = z/L_D \), \( \xi = \sqrt{2}Ax/w_0 \), and \( u(\xi, \zeta) = (k|n_2|L_D/n_0)^{1/2} A(\xi, \zeta) \) are longitudinal and transverse coordinates and the field amplitude, respectively, in terms of Kerr coefficient \( n_2 \), unscaled variables \( z \) and \( x \), and field envelope \( A \) defined by \( E(x, z) = A(x, z) \exp(ikz) \). For simplicity, a uniform-background field \( u_0 \) is assumed. A general dark solution of Eq. (1) is then found to be

\[
u(\xi, \zeta) = u_0(A \tanh \Theta + iF)
\]

\[
\times \exp \left[ i \left( 1 - 4\kappa u_0^2 \right)^{1/2} \left( -V \xi + \frac{\zeta}{2\kappa} \right) \right]
\]

\[
\times \exp \left( -i \frac{\zeta}{2\kappa} \right),
\]

where

\[
\Theta = \frac{u_0 A(\xi + W \zeta)}{(1 + 2\kappa W^2)^{1/2}}
\]

and

\[
W = \frac{V - V_0}{1 + 2\kappa V V_0}
\]

is a net transverse velocity involving \( V \) (from the choice of reference direction) and \( V_0 \) (a gray-soliton component), given by

\[
V_0 = \frac{u_0 F}{\left[ 1 - (2 + F^2)2\kappa u_0^2 \right]^{1/2}}.
\]

F and \( A \) are real constants, where \( F = \pm (1 - A^2)^{1/2} \).
whereas $|F| > 0$ yields gray solitons. In the paraxial limit, the nonlinear Schrödinger equation (NLS) dark soliton \cite{11,12,13} is obtained:

$$u(\xi, \zeta) = u_0(A \tanh \Theta + iF)\exp(-iV\xi)$$

$$\times \exp\left(-iu_0^2\xi - i\frac{1}{2}V^2\zeta\right),$$

(6)

where

$$\Theta = u_0A[\xi + (V - Fu_0)\zeta].$$

(7)

A particular Helmholtz dark-soliton solution for which $V = 0$ and $F = 0$ was reported earlier. \cite{14,15} Interestingly, a simple ansatz approach is insufficient for determining the complete general solution presented here. Instead, we have also used geometrical considerations and invariance relations for Helmholtz solutions when the axes are rotated:

$$\xi = \frac{\xi' + V\zeta'}{1 + 2kV^2g^{1/2}}, \quad \zeta = \frac{-2kV\xi' + \zeta'}{1 + 2kV^2g^{1/2}},$$

$$u(\xi, \zeta) = \exp\left(i\frac{V\zeta'}{(1 + 2kV^2g)^{1/2}} \left[1 - \frac{1}{(1 + 2kV^2g)^{1/2}}\right]^{1/2}\right)u'(\xi', \zeta'),$$

(8)

where $\zeta' / H^{1/2}$ is defined by choice of reference frame (see Fig. 1).

The phase period of Helmholtz dark solitons is governed by the longitudinal wave number that is given by two factors: exp($-i\xi / 2k$), which is due to the forward reference frame, and projection of the nonlinear correction factor $(1 - 4k\zeta u_0^2g^{1/2})$ onto the $\zeta$ axis. The transverse velocity of paraxial dark solitons is $V - Fu_0$. However, a more accurate description involves velocity summation in the unscaled coordinate system, Eq. (4), and modifications of the intrinsic gray-soliton velocity, Eq. (5).

Nonzero $W$ corresponds to off-axis propagation. Geometrical considerations then imply that the beam width projected onto the transverse axis should increase (a feature absent from paraxial theory, in which this width is constant). In fact, the inverse soliton width is given by

$$\xi_0^{-1} = \frac{u_0A}{(1 + 2kW^2g)^{1/2}}.$$  

(9)

The beam width enlargement factor can also be written as $(1 + 2kW^2g)^{1/2} = \sec(\theta - \theta_0)$, where $\theta_0 = \sec^{-1}(1 + 2kW^2g^{1/2})$ is the angle associated with $V_0$ and $\theta = \sec^{-1}(1 + 2kV^2g^{1/2})$ is defined by choice of reference frame (see Fig. 1).

Paraxial dark solitons exist for arbitrary values of background intensity. \cite{11,12,13} Helmholtz black solitons exist only for $4k\zeta u_0^2 < 1$, which corresponds to $|2n_2I| < n_0$, where $I = |E_0|^2$ is the unscaled background intensity, and when the size of the nonlinear phase shift is less than the linear contribution. The refractive index thus remains positive (a condition implicit in the paraxial NLS that appears explicitly in the NHE solution). Paraxial gray solitons exist for any nonzero $|F| < 1$, whereas Helmholtz gray solitons have a more limited range of $F$, given by $0 < |F| < |F|_{\text{max}} = (1 - 4\kappa u_0^2)^{1/2}/(2\kappa u_0^2g^{1/2})$, where $\theta_0 = \pm \pi/2$ for $|F| = |F|_{\text{max}}$, revealing the physical limit imposed on the largest possible transverse velocity.

Helmholtz dark solitons can be studied by use of recently developed numerical techniques; analysis uncovered intrinsic limitations of traditional approaches. \cite{14} To explore whether Helmholtz dark solitons are spontaneously created from an initial field profile that does not correspond to an exact soliton, we first consider the initial condition

$$u(\xi, 0) = u_0 \tanh(u_0 a\xi).$$

(10)

Parameter $a$ controls the inverse width of the initial profile. The generation of $2N_0 + 1$ paraxial solitons is expected during evolution governed by the NLS. \cite{11,12} (Here $N_0$ is the largest integer that satisfies $N_0 < 1/a$.)

NLS and NHE simulations have been carried out for $1 \leq U_0 \leq 5, 0 < a \leq 1$, and $\kappa = 0.001$. Figure 2 shows...
In a paraxial framework, a single black soliton results 

\[ u(\xi, 0) = u_0 \tanh(u_0 \xi) \exp(-iS_0 \xi). \tag{11} \]

In a paraxial framework, a single black soliton results 

that has transverse velocity \( S_0 \). For \( \kappa u_0^2 \ll 1 \), the NHE evolution can be shown to be equivalent to the propagation of an initially perturbed (reduced width) on-axis NLS black soliton. Inverse scattering techniques,\(^{11-15}\) then, implies the generation of only one Helmholtz soliton. Figure 4 shows the evolving beam widths for \( u_0 = 1, \kappa = 0.001 \), and three values of \( S_0 \) that correspond to propagation angles of \( \theta = \tan^{-1}(\sqrt{2 \kappa V}) = 12.9^\circ, 26.6^\circ, 42.1^\circ \), respectively. The predicted asymptotic values of the Helmholtz beam width are given by \((1 + 2 \kappa V^2)^{1/2} \), where 

\[ V = S_0/(1 - 2 \kappa S_0^2)^{1/2} \] 

when \( \kappa u_0^2 \ll 1 \). Whereas similarly perturbed Helmholtz bright solitons undergo large oscillations over long propagation distances (\( \zeta > 50 \)),\(^6\) dark beams are found to exhibit a surprisingly fast convergence to the asymptotic solutions.

The Helmholtz solitons discussed here are likely to lead to a new class of soliton solution, modified by the same Helmholtz-type correction, appropriate to generalized nonlinearities, higher dimensions (e.g., stripes, rings, vortices), coupled modes (e.g., interactions, vector solitons) and other soliton wave equations.\(^{1,13}\)

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**References**